



The Generalization of Combination Number

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Abstract

In this paper, the concepts of combination number, factorial and Euler function are generalized, and the concepts of generalized combination number, generalized Euler function and m factorial of n are proposed. Several typical Identities and congruence of generalized combination number are proved, and the calculation formulas of generalized Euler function are derived. Using the m factorial of n , the congruence formula of the modular prime of the original factorial is generalized.

Subject Areas

Integral Equation, Number Theory

Keywords

Generalization of Combination Number, Factorial Generalization, Euler Function Generalization

1. Description and Introduction

1.1. Description

The following defined symbols will be used in this article:

- 1) $\binom{n}{k}$: Common combination number.
- 2) $n!$: n factorial.
- 3) $\varphi(n)$: Euler's function.
- 4) $\binom{n}{k}_m$: Generalized combination number.
- 5) $(n!)_m$: m factorial of n (generalized factorial).
- 6) $\varphi_m(n)$: generalized Euler function is of fundamental.

1.2. Introduction

Combinatorial numbers Importance in number theory [1], in the paper, we generalize combinatorial number, Factorial and Euler functions, and prove the identities and congruence of several Typical generalized combinatorial numbers; by using m factorial of n , we generalize the conclusion of

$$\left(\frac{p-1}{2}\right)!^2 \equiv \begin{cases} -1 \pmod{p} & p \equiv 1 \pmod{4} \\ 1 \pmod{p} & p \equiv -1 \pmod{4} \end{cases}$$

p is a odd prime.

Generalized factorials and generalized Euler functions are also used in the derivation of properties and congruence of generalized combinatorial numbers.

In this paper, a conjecture related to generalize combinatorial number is presented.

2. Definition

1) Generalization of combinational number: $\binom{n}{k}_m$

Let $n, m, k \in \mathbb{Z}^+$, then
$$\binom{n}{k}_m = \frac{\prod_{\substack{i=n-k+1 \\ (i,m)=1}}^n i}{\prod_{\substack{j=1 \\ (j,m)=1}}^k j}.$$

Example:
$$\binom{7}{13}_{22} = \frac{(-5)(-3)(-1) \times 1 \times 3 \times 5 \times 7}{1 \times 3 \times 5 \times 7 \times 9 \times 13} = -\frac{5}{39},$$

$$\binom{11}{7}_{15} = \frac{7 \times 8 \times 11}{1 \times 2 \times 4 \times 7} = 11.$$

$$\binom{12}{7}_{13} = \frac{6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7} = 792, \quad \binom{18}{6}_{18} = \frac{13 \times 17}{5 \times 1} = \frac{221}{5}.$$

2) m factorial of n : $(n!)_m$

If $n \geq 0, m > 0$, then
$$(n!)_m = \prod_{\substack{i=1 \\ (i,m)=1}}^n i.$$

Appoint: $(0!)_m = 1.$

Example: $(13!)_{21} = 1 \times 2 \times 4 \times 5 \times 8 \times 10 \times 11 \times 13 = 457600,$

$(9!)_{15} = 1 \times 2 \times 4 \times 7 \times 8 = 448.$

$$(4!)_1 = 1 \times 2 \times 3 \times 4 = 24.$$

3) $n \geq 0, m \in \mathbb{Z}^+, \varphi_m(n)$ is the number of numbers that are not greater than n and are coprime with m . Appoint: $\varphi_m(0) = 0.$

Example: $\varphi_{15}(11) = 6, \varphi_{21}(8) = 5, \varphi_{35}(35) = 24, \varphi_7(12) = 11.$

3. Properties of Generalized Combination Numbers

1) $n, k \in \mathbb{N}^+$, then:
$$\binom{n}{k}_1 = \binom{n}{k}.$$

Proof. According to the generalized combination number definition.

$$2) \ n \geq k > 0, \text{ then } \binom{n}{k}_m = \frac{(n!)_m}{(k!)_m ((n-k)!)_m}.$$

Proof. According to the definition of generalized combination number and m factorial of n .

$$\binom{n}{k}_m = \frac{\prod_{\substack{i=n-k+1 \\ (i,m)=1}}^n i}{\prod_{\substack{j=1 \\ (j,m)=1}}^k j} \times \frac{((n-k)!)_m}{((n-k)!)_m} = \frac{(n!)_m}{(k!)_m ((n-k)!)_m}.$$

$$3) \text{ When: } 0 < n < k, \text{ then: } \binom{n}{k}_m = \frac{(-1)^{\varphi_m(k-n-1)} (n!)_m ((k-n-1)!)_m}{(k!)_m}.$$

Proof. According to the definition of generalized combination number and m factorial of n

$$\binom{n}{k}_m = \frac{(n!)_m (-1) \cdots -(k-n-1)}{(k!)_m} = \frac{(-1)^{\varphi_m(k-n-1)} (n!)_m ((k-n-1)!)_m}{(k!)_m}.$$

4) Let $n > k > 0$, then

$$\binom{n}{k}_m = \binom{n}{n-k}_m.$$

Proof. by nature 3.2

$$\binom{n}{k}_m = \frac{(n!)_m}{(k!)_m ((n-k)!)_m}, \binom{n}{n-k}_m = \frac{(n!)_m}{((n-k)!)_m (k!)_m}.$$

$$\text{therefore: } \binom{n}{k}_m = \binom{n}{n-k}_m.$$

5) If $n > 1, k > 1, (nk, m) = 1$, then

$$\binom{n}{k}_m = \frac{n}{k} \binom{n-1}{k-1}_m.$$

Proof. When $n > k > 1$, by nature 3.2

$$\binom{n}{k}_m = \frac{(n!)_m}{(k!)_m ((n-k)!)_m}, \binom{n-1}{k-1}_m = \frac{((n-1)!)_m}{((k-1)!)_m ((n-k)!)_m}.$$

Because: $(nk, m) = 1$, therefore

$$\frac{n}{k} \binom{n-1}{k-1}_m = \frac{n}{k} \times \frac{((n-1)!)_m}{((k-1)!)_m ((n-k)!)_m} = \binom{n}{k}_m$$

$$\text{When } n = k, \binom{n}{n}_m = \frac{n}{n} \binom{n-1}{n-1}_m = 1.$$

When: $1 < n < k$, by nature 3.3

$$\binom{n}{k}_m = \frac{(-1)^{\varphi_m(k-n-1)} (n!)_m ((k-n-1)!_m)}{(k!)_m},$$

$$\binom{n-1}{k-1}_m = \frac{(-1)^{\varphi_m(k-n-1)} ((n-1)!_m ((k-n-1)!_m)}{((k-1)!_m)}.$$

Because: $(nk, m) = 1$, therefore $\binom{n}{k}_m = \frac{n}{k} \binom{n-1}{k-1}_m$.

6) If $n > r > 0, k > r > 0$, then

$$\binom{n}{k}_m \times \binom{k}{r}_m = \binom{n}{r}_m \times \binom{n-r}{k-r}_m.$$

Proof. When $n > k > r > 0$, by nature 3.2

$$\begin{aligned} \binom{n}{k}_m \binom{k}{r}_m &= \frac{(n!)_m}{(k!)_m ((n-k)!_m)} \times \frac{(k!)_m}{(r!)_m ((k-r)!_m)} \\ &= \frac{(n!)_m}{(r!)_m ((n-k)!_m ((k-r)!_m)}, \\ \binom{n}{r}_m \binom{n-r}{k-r}_m &= \frac{(n!)_m}{(r!)_m ((n-r)!_m)} \times \frac{((n-r)!_m)}{((k-r)!_m ((n-k)!_m)} \\ &= \frac{(n!)_m}{(r!)_m ((n-k)!_m ((k-r)!_m)}. \end{aligned}$$

therefore: $\binom{n}{k}_m \binom{k}{r}_m = \binom{n}{r}_m \binom{n-r}{k-r}_m$.

When: $n = k > r > 0, \binom{n}{n}_m = \frac{n}{n} \binom{n-r}{n-r}_m = 1$.

therefore: $\binom{n}{k}_m \binom{k}{r}_m = \binom{n}{r}_m \binom{n-r}{k-r}_m$.

When: $0 < r < n < k$, by nature 3.3

$$\begin{aligned} \binom{n}{k}_m \binom{k}{r}_m &= \frac{(-1)^{\varphi_m(k-n-1)} (n!)_m ((k-n-1)!_m)}{(k!)_m} \times \frac{(k!)_m}{(r!)_m ((k-r)!_m)} \\ &= \frac{(-1)^{\varphi_m(k-n-1)} (n!)_m ((k-n-1)!_m)}{(r!)_m ((k-r)!_m)}, \\ \binom{n}{r}_m \binom{n-r}{k-r}_m &= \frac{(n!)_m}{(r!)_m ((n-r)!_m)} \times \frac{(-1)^{\varphi_m(k-n-1)} ((n-r)!_m ((k-n-1)!_m)}{((k-r)!_m)} \\ &= \frac{(-1)^{\varphi_m(k-n-1)} (n!)_m ((k-n-1)!_m)}{(r!)_m ((k-r)!_m)}. \end{aligned}$$

therefore when $n > r > 0, k > r > 0, \binom{n}{k}_m \binom{k}{r}_m = \binom{n}{r}_m \binom{n-r}{k-r}_m$.

7) Let $n > k > 0, (k+1, m) = 1, (n-k, m) = 1, (n+1, m) = 1$, then

$$\binom{n}{k}_m + \binom{n}{k+1}_m = \binom{n+1}{k+1}_m.$$

Proof. By nature 3.2

$$\binom{n}{k}_m = \frac{(n!)_m}{(k!)_m ((n-k)!)_m} \times \frac{k+1}{k+1},$$

$$\binom{n}{k+1}_m = \frac{(n!)_m}{((k+1)!)_m ((n-(k+1))!)_m} \times \frac{n-k}{n-k}.$$

$$\binom{n+1}{k+1}_m = \frac{((n+1)!)_m}{((k+1)!)_m ((n-k)!)_m}. \text{ because: } (k+1, m) = 1, (n-k, m) = 1, (n+1, m) = 1$$

$$\binom{n}{k}_m + \binom{n}{k+1}_m = \frac{(n!)_m \times (n+1)}{((k+1)!)_m ((n-k)!)_m}.$$

$$\text{therefore: } \binom{n}{k}_m + \binom{n}{k+1}_m = \binom{n+1}{k+1}_m.$$

8) Let $1 < k < n$, $(k, m) = 1$, $(n-k+1, m) = 1$, then

$$\binom{n}{k-1}_m = \binom{n}{k}_m \times \frac{k}{n-k+1}.$$

Proof. By nature 3.2

$$\binom{n}{k-1}_m = \frac{(n!)_m}{((k-1)!)_m ((n-k+1)!)_m}, \binom{n}{k}_m = \frac{(n!)_m}{(k!)_m ((n-k)!)_m}.$$

Because $(k, m) = 1$, $(n-k+1, m) = 1$, therefore

$$\begin{aligned} \binom{n}{k}_m \times \frac{k}{n-k+1} &= \frac{(n!)_m}{(k!)_m ((n-k)!)_m} \times \frac{k}{n-k+1} \\ &= \frac{(n!)_m}{((k-1)!)_m ((n-k+1)!)_m} = \binom{n}{k-1}_m. \end{aligned}$$

4. Congruence of Generalized Combination Numbers

$$1) \text{ If } m > k > 1, (m, k) = 1, \text{ then } \binom{m}{k}_m \equiv \frac{(-1)^{\phi_m(k)-1}}{k} \pmod{m};$$

$$\text{If } k > m > 1, (m, k) = 1, \text{ then } \binom{m}{k}_m \equiv \frac{(-1)^{\phi_m(k-m-1)}}{k} \pmod{m};$$

Proof. When $m > k > 1$, by nature 3.2

$$\binom{m}{k}_m = \frac{(m!)_m}{(k!)_m ((m-k)!)_m} \equiv \frac{(-1)^{\phi_m(k)-1} \times ((k-1)!)_m}{k \times ((k-1)!)_m} \pmod{m}, \text{ because}$$

$$(((k-1)!)_m, m) = 1, \text{ therefore } \binom{m}{k}_m \equiv \frac{(-1)^{\phi_m(k)-1}}{k} \pmod{m}.$$

When $k > m > 1$, by nature 3.3

$$\begin{aligned} \binom{m}{k}_m &= \frac{(-1)^{\varphi_m(k-m-1)} (m!)_m ((k-m-1)!)_m}{(k!)_m} \\ &\equiv \frac{(-1)^{\varphi_m(k-m-1)} (m!)_m ((k-m-1)!)_m}{k \times (m!)_m \times ((k-m-1)!)_m} \pmod{m} \end{aligned}$$

Because $((m!)_m \times ((k-m-1)!)_m, m) = 1$, therefore

$$\binom{n}{k}_m \equiv \frac{(-1)^{\varphi_m(k-m-1)}}{k} \pmod{m}.$$

2) If $m > 1$, $(k, m) = 1$, then $\binom{m-1}{k}_m \equiv (-1)^{\varphi_m(k)} \pmod{m}$,

$$\binom{m-1}{m-k}_m \equiv (-1)^{\varphi_m(k)+1} \pmod{m}.$$

Proof. $(k!)_m \binom{m-1}{k}_m = \prod_{\substack{i=m-k \\ (i,m)=1}}^{m-1} i \equiv \prod_{\substack{j=-k \\ (j,m)=1}}^{-1} j \equiv (-1)^{\varphi_m(k)} (k!)_m \pmod{m}$

$$m \left| (k!)_m \left[\binom{m-1}{k}_m - (-1)^{\varphi_m(k)} \right], \text{ because } (m, (k!)_m) = 1, m \left| \binom{m-1}{k}_m - (-1)^{\varphi_m(k)}, \right.$$

$$\text{therefore } \binom{m-1}{k}_m \equiv (-1)^{\varphi_m(k)} \pmod{m}.$$

$$\text{similarly } \binom{m-1}{m-k}_m \equiv (-1)^{\varphi_m(k)+1} \pmod{m}.$$

3) If $m > 1$ is a even number, $(k, m) = 1$, then $\binom{k}{m-1}_m \equiv 1 \pmod{m}$.

Proof. When $m = 2$, according to the generalized combination number definition, k is a odd number, therefore $\binom{k}{m-1}_m = \binom{k}{1}_2 = k \equiv 1 \pmod{2}$.

If $m > 2$ is a even number, $k \leq m-1$,

$$\begin{aligned} \binom{k}{m-1}_m &= \frac{(-1)^{\varphi_m(m-k-2)} (k!)_m ((m-k-2)!)_m}{((m-1)!)_m} \\ &\equiv \frac{(-1)^{\varphi_m(m-k-2)} (k!)_m ((m-k-2)!)_m}{(-1)^{\varphi_m(m-k-2)} ((m-k-2)!)_m (k!)_m} \pmod{m} \end{aligned}$$

$$\text{therefore: } \binom{k}{m-1}_m \equiv 1 \pmod{m}.$$

When $m > 2$ is a even number, $k > m-1$, by nature 3.2

$$\begin{aligned} ((m-1)!)_m \binom{k}{m-1}_m &= \frac{(k!)_m}{((k-(m-1)!)_m)} \\ &= \frac{((m-1)!)_m ((k-(m-1)!)_m)}{((k-(m-1)!)_m)} \\ &\equiv ((m-1)!)_m \pmod{m}. \end{aligned}$$

Because $\left(\left((m-1)!\right)_m, m\right) = 1$, therefore $\binom{k}{m-1}_m \equiv 1 \pmod{m}$.

4) $m > 1$, then $\sum_{\substack{k=1 \\ (k,m)=1}}^{m-1} \binom{m-1}{k}_m \equiv 0 \pmod{m}$.

Proof. by congruence 4.2

5) $m > 1$ is a even number, then $\sum_{\substack{k=1 \\ (k,m)=1}}^{m-1} \binom{k}{m-1}_m \equiv \varphi_m(m) = \varphi(m) \pmod{m}$.

Proof. By congruence 4.3.

6) If $m > 1$ is a odd number, then $\binom{\frac{m-1}{2}}{m-1}_m \equiv 2 \pmod{m}$.

Proof.

$$\begin{aligned} \left(\left(\frac{m-1}{2}\right)_m, m\right) &= \left(\left(\frac{m-1}{2}\right)!, m\right) \times (-1) \times \dots \times \left(\frac{m-1}{2} - m + 2\right) \\ &\equiv \frac{(-1)^{\varphi(m)} \left(\left(\frac{m-1}{2}\right)!\right)_m \times 2}{-(m-1)} \pmod{m} \end{aligned}$$

Because $\left(\left(\frac{m-1}{2}\right)_m, m\right) = 1$, $\varphi(m)$ is a even, $-(m-1) \equiv 1 \pmod{m}$

$$\text{therefore } \binom{\frac{m-1}{2}}{m-1}_m \equiv 2 \pmod{m}.$$

5. The Method of Calculating the Value of the Generalized Euler Function

Let $m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, $n > 1$, then

$$\varphi_m(n) = n - \sum_{i=1}^r \left[\frac{n}{p_i} \right] + \sum_{1 \leq i < j \leq r} \left[\frac{n}{p_i p_j} \right] - \dots + (-1)^r \left[\frac{n}{p_1 p_2 \dots p_r} \right].$$

Proof. Because $\varphi(m) = m - \sum_{i=1}^r \frac{m}{p_i} + \sum_{1 \leq i < j \leq r} \frac{m}{p_i p_j} - \dots + (-1)^r \frac{m}{p_1 \dots p_r}$ [2].

$$\text{therefore: } \varphi_m(n) = n - \sum_{i=1}^r \left[\frac{n}{p_i} \right] + \sum_{1 \leq i < j \leq r} \left[\frac{n}{p_i p_j} \right] - \dots + (-1)^r \left[\frac{n}{p_1 p_2 \dots p_r} \right].$$

6. Lemma

Lemma: Let $m > 0$, $a_1, a_2, \dots, a_{\varphi(m)}$ is all the numbers in m that are coprime

with m , then $a_1 a_2 \dots a_{\varphi(m)} \equiv \begin{cases} -1 \pmod{m}, & m = 2, 4, p^\alpha, 2p^\alpha \\ 1 \pmod{m}, & m \neq 2, 4, p^\alpha, 2p^\alpha \end{cases}$. Among: p is a odd prime [3].

7. A Theorem for Generalized Factorials

Theorem:

1) p is a odd prime, $p \equiv 1 \pmod{4}$, $m = p^r$, then

$$\left(\left(\frac{m-1}{2}\right)_m\right)^2 \equiv -1 \pmod{m}.$$

2) p is a odd prime, $p \equiv 1 \pmod{4}$, $m \neq p^r$, $m > 1$ is a odd then

$$\left(\left(\frac{m-1}{2}\right)_m\right)^2 \equiv 1 \pmod{m}.$$

$$3) \ m > 1 \text{ is a odd, then } \left(\frac{m-1}{2}\right)_m \equiv \frac{\prod_{\substack{i=\frac{m+1}{2} \\ (i,m)=1}}^{m-1} i}{\prod_{\substack{j=1 \\ (j,m)=1}}^{\frac{m-1}{2}} j} \equiv (-1)^{\frac{\varphi(m)}{2}} \pmod{m}.$$

Proof. When $m > 1$.

$$\begin{aligned} ((m-1)!)_m &= 1 \times 2 \times \dots \times \left(\frac{m-1}{2}\right) \times (m-1)(m-2) \times \dots \times \left(m - \frac{m-1}{2}\right) \\ &\equiv 1 \times 2 \times \dots \times \left(\frac{m-1}{2}\right) \times (-1)(-2) \times \dots \times \left(-\frac{m-1}{2}\right) \\ &\equiv (-1)^{\frac{\varphi(m)}{2}} \left(\left(\frac{m-1}{2}\right)_m\right)^2 \pmod{m}. \end{aligned}$$

$$\text{Instant: } ((m-1)!)_m \equiv (-1)^{\frac{\varphi(m)}{2}} \left(\left(\frac{m-1}{2}\right)_m\right)^2 \pmod{m}. \tag{1}$$

When $p \equiv 1 \pmod{4}$, $m = p^r$, $\frac{\varphi(m)}{2}$ is a even; When $p \equiv -1 \pmod{4}$, $m = p^r$, $\frac{\varphi(m)}{2}$ is a odd. When $m \neq p^r$ is a odd, $\frac{\varphi(m)}{2}$ is a even, according to the lemma: $a_1 a_2 \dots a_{\varphi(m)} = ((m-1)!)_m \equiv \begin{cases} -1 \pmod{m} : m = 2, 4, p^\alpha, 2p^\alpha \\ 1 \pmod{m} : m \neq 2, 4, p^\alpha, 2p^\alpha \end{cases}$ p is a

odd prime, $p \equiv 1 \pmod{4}$, $m = p^r$, then $\left(\left(\frac{m-1}{2}\right)_m\right)^2 \equiv -1 \pmod{m}$.

$p \equiv 1 \pmod{4}$, $m \neq p^r$, m is a odd, then $\left(\left(\frac{m-1}{2}\right)_m\right)^2 \equiv 1 \pmod{m}$.

When $m > 1$ is a odd, according to (1):

$$\left(\frac{m-1}{2}\right)_m \equiv \frac{\prod_{\substack{i=\frac{m+1}{2} \\ (i,m)=1}}^{m-1} i}{\prod_{\substack{j=1 \\ (j,m)=1}}^{\frac{m-1}{2}} j} \equiv (-1)^{\frac{\varphi(m)}{2}} \pmod{m}.$$

8. A Guess

After a lot of calculation and verification, the following conjecture is proposed:

Let $m > 1$, $m \equiv 1 \pmod{2}$, $(k, m) = 1$, $\sigma_k = k$ or $\sigma_k = m - k$, make

$$\binom{m-1}{\sigma_k}_m \equiv (-1)^{k+1} \pmod{m}$$

$$\text{then } \frac{2^{\varphi(m)} - 1}{m} \equiv \frac{1}{2} \sum_{\substack{k=1 \\ (k,m)=1}}^{m-1} \frac{1}{k} \times \binom{m-1}{\sigma_k}_m \pmod{m}.$$

If this conjecture is true, it will be an important characterization of the generalized Combinatorial numbers.

9. Concluding Remarks

According to $\binom{n}{k}_1 = \binom{n}{k}$, we can know that the total generalized combinatorial

Number set is a larger number set than the original combinatorial number set, which is a generalization of the original combinatorial number set? Of course, the generalized combinatorial number is also a new number, for which there must be many properties not yet understood, and some applications are still to be discovered.

Conflicts of Interest

The authors declare no conflicts of interest.

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